JOURNAL OF ALGEBRA **210**, 361–363 (1998)
ARTICLE NO. JA987554

On a Theorem of Broline

Keresztély Corrádi

Department of Comp. Technology, Eötvös University Budapest, H-1088 Budapest, Hungary

and

László Héthelyi

Department of Algebra, Technical University of Budapest, H-1111 Budapest, Hungary

Communicated by Walter Feit

Received March 27, 1998

1. INTRODUCTION

Broline has proved that if $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$, $K = \mathrm{Ker}\,\chi$, $F/K = F(G/K)$, and F is not nilpotent then there exists a $\psi \in \mathrm{Irr}_{\mathbb{C}}(G)$ such that $\mathrm{Ker}\,\psi < K$. Here $F(G)$ stands for the Fitting subgroup of the finite group G. From his proof [2, p. 211] it follows that ψ can be chosen so that $\psi(1) \geq \chi(1)$ holds as well, and the theorem with the above extension remains valid if F is an arbitrary, nonnilpotent normal subgroup of G for which F/K is nilpotent.

In the present paper we shall show that a somewhat sharpened result can be proved if, in addition, we suppose that $(|F{:}K|, \chi(1)) = 1$, namely the above ψ can be chosen so that $\psi(1) > \chi(1)$. So the main result of the paper may be stated as our

THEOREM. *Let G be a finite group, $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$, $K = \mathrm{Ker}\,\chi$, $K \leq F \lhd G$ such that F/K is nilpotent. If $(|F : K|, \chi(1)) = 1$ and F is nonnilpotent, then there exists a $\psi \in \mathrm{Irr}_{\mathbb{C}}(G)$ such that $\psi(1) > \chi(1)$ and $\mathrm{Ker}\,\psi < K$.*

We mention that the case $K = F$ of the theorem gives the well-known result of Brolin and Garrison (see Theorem 12.19 in [2]) whose assertion will be used in our proof. Before concluding this section we emphasize that all groups considered in the paper are finite.

0021-8693/98 $25.00

2. PROOF OF THE THEOREM

The proof is by contradiction. Let the quadruple (G, χ, K, F) be a counterexample for which $|G| + |F|$ is minimal. By our choice there exists a $q \in \pi(F)$ and a $Q \in \mathrm{Syl}_q(F)$ such that $Q \ntrianglelefteq F$, $F = KQ$, and $Q \nleq K$. Use Theorem 12.19 in [2] which asserts that K is nilpotent. Besides according to our hypothesis $q \nmid \chi(1)$ holds. Since $Q \ntrianglelefteq G$ there exists a maximal subgroup H in G with $N_G(Q) \leq H$. By the Frattini argument $G = FN_G(Q) = KN_G(Q)$. Hence $G = HK$. Set $D = H \cap K$. We deduce a contradiction in three steps.

Step 1: $D \neq 1$. Suppose that $D = 1$. Since H is a maximal subgroup of G, K is an elementary abelian p-group for some prime. By the proof of Theorem 12.24 in [2] one can choose an $\eta \in \mathrm{Irr}_{\mathbb{C}}(G)$ such that $(\eta, \vartheta^G)_G \neq 0$, where $\vartheta = \chi \restriction H$ and $K \nleq \mathrm{Ker}\, \eta$. It is immediately seen that $\vartheta \in \mathrm{Irr}_{\mathbb{C}}(H)$ and that $\mathrm{Ker}\, \eta = 1$. For such an η by Clifford's theorem

$$\eta \upharpoonright F = e \sum_{1 \leq j \leq t} \tau_j,$$

where $\tau_j \in \mathrm{Irr}_{\mathbb{C}}(F)$, $1 \leq j \leq t$, and $\tau_j(1) = \tau_1(1)$, $1 \leq j \leq t$. As $\mathrm{Ker}\, \eta = 1$ and $F' \neq 1$, $\tau_1(1) \neq 1$. On the other hand, as $F = K(F \cap H)$ and $K' = 1$, so by Ito's theorem we have $\kappa(1) \,|\, |F \cap H|$ and consequently $\kappa(1) \,|\, |Q|$ for any $\kappa \in \mathrm{Irr}_{\mathbb{C}}(F)$. Especially $q \mid \tau_j(1)$, $1 \leq j \leq t$, and so $q \mid \eta(1)$. Since $\eta(1) = \vartheta(1)$ we obtain a contradiction.

Step 2: $F \cap H$ is nilpotent. Suppose the contrary. Observe that $\chi \restriction H = \vartheta \in \mathrm{Irr}_{\mathbb{C}}(H)$. The quadruple $(H, \vartheta, D, F \cap H)$ satisfies the conditions of our theorem. Since $F = (F \cap H)K$ and since $D \triangleleft H$,

$$(F \cap H)/D = (F \cap H)/(K \cap H) \simeq K(F \cap H)/K \simeq F/K$$

is a consequence. So we have that there exists a $\lambda \in \mathrm{Irr}_{\mathbb{C}}(H)$ such that $\lambda(1) > \chi(1)$ and $\mathrm{Ker}\, \lambda < \mathrm{Ker}\, \vartheta = D$. Use the choice of our quadruple (G, χ, K, F). Let now $\eta \in \mathrm{Irr}_{\mathbb{C}}(G)$ such that $(\eta \restriction H, \lambda)_H \neq 0$. Then $\eta(1) \geq \lambda(1) > \chi(1)$ and $H \cap \mathrm{Ker}\, \eta < D$. If $\mathrm{Ker}\, \eta \leq H$ holds then η satisfies the conclusions of the theorem. Since this is not the case, we may suppose that $T = \mathrm{Ker}\, \eta \nleq H$. Hence $G = HT$. Set $U = TD$. Then $U \triangleleft G$, and $U \cap H = TD \cap H = (T \cap H)D = D$. Since $T \cap H < D$ we have $T < U$. Thus

$$G/U = HU/U \simeq H/U \cap H = H/D \simeq G/K. \qquad (*)$$

Let $\zeta \in \mathrm{Irr}_{\mathbb{C}}(G)$ such that $\mathrm{Ker}\, \zeta = U$ and $\zeta \restriction H = \vartheta$. By $(*)$ ζ exists. Then $(\vartheta^G, \zeta)_G = (\vartheta, \vartheta)_H = 1$. So ζ is a constituent of ϑ^G. If $U \neq K$ then by the proof of Theorem 12.24 in [2] $\mathrm{Ker}\, \zeta < K$, which contradicts $(*)$.

So $U = K$ (and $\zeta = \chi$ satisfies the above conditions) thus $T < K$, which shows that η satisfies the conditions of the theorem. This contradicts the choice of the quadruple (G, χ, K, F). This forces that $F \cap H$ is nilpotent.

Step 3. We show that the nilpotency of $F \cap H$ leads to a contradiction. Since $G = HK$ and since K is nilpotent, $D = H \cap K \triangleleft G$. Hence $D \cap Z(K) \neq 1$. Set $A = D \cap Z(K)$. As $A \triangleleft G$ and as $A \leq F \cap H$, the nilpotency of $F \cap H$ gives that $A \cap Z(F \cap H) = L \neq 1$. By $F = (F \cap H)K$ we deduce that $L \leq Z(F)$. So $Z(F) \neq 1$, and $D \cap Z(F) \neq 1$. Set $B = D \cap Z(F)$, then $1 \neq B \triangleleft G$. Define $\overline{G}$ by $\overline{G} = G/B$ and let $\overline{K}$, $\overline{F}$, and $\overline{Q}$ be the images of K, F, and Q in $\overline{G}$, respectively. The $\overline{Q} \in \mathrm{Syl}_q(\overline{F})$ and as $\overline{Q} \ntriangleleft \overline{F}$ the quadruple $(\overline{G}, \overline{\chi}, \overline{K}, \overline{F})$ satisfies the conditions of the theorem. So by the choice of the quadruple (G, χ, K, F) we have a $\overline{\psi} \in \mathrm{Irr}_{\mathbb{C}}(\overline{G})$ such that $\overline{\psi}(1) > \overline{\chi}(1)$ and $\mathrm{Ker}\,\overline{\psi} < \overline{K}$. So the $\psi \in \mathrm{Irr}_{\mathbb{C}}(G)$ which is associated to $\overline{\psi}$ in the natural way satisfies the conclusions of the Theorem. This contradicts the existence of the quadruple (G, χ, K, F).

REFERENCES

1. D. M. Broline, Nilpotence and character kernels, *J. Algebra* **45** (1977), 83–87.
2. I. M. Isaacs, "Character Theory of Finite Groups," Academic Press, New York, 1976.